

# Midterm COMP 2804

February 27, 2014

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- This is a closed-book exam.
- Calculators are not allowed.

**Marking scheme:** Each of the 17 questions is worth 1 mark.

- Newton:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ .

1. On a table, you see three types of fruit: apples, bananas, and oranges. There are  $m \geq 2$  apples,  $n \geq 2$  bananas, and  $k \geq 2$  oranges. How many ways are there to choose 7 pieces of fruit, if you must take at least two pieces of each type?

- (a)  $\binom{m+n+k}{7} - (m+n+k)$
- (b)  $\binom{m+n+k}{7} - \binom{m}{2} - \binom{n}{2} - \binom{k}{2}$
- (c)  $\binom{m}{3}\binom{n}{2}\binom{k}{2} + \binom{m}{2}\binom{n}{3}\binom{k}{2} + \binom{m}{2}\binom{n}{2}\binom{k}{3}$
- (d)  $\binom{m}{2}\binom{n}{2}\binom{k}{2}(m+n+k)$

2. Consider 9 boys and 15 girls. How many ways are there to arrange these 24 people on a line if all boys stand next to each other and all girls stand next to each other?

- (a)  $\frac{24!}{9!15!}$
- (b)  $\binom{24}{9}(9!)(15!)$
- (c)  $(9!)(15!)$
- (d)  $2(9!)(15!)$

3. Let  $S$  be a set of size 37, and let  $x$ ,  $y$ , and  $z$  be three distinct elements of  $S$ . How many subsets of  $S$  are there that contain  $x$  and  $y$ , but do not contain  $z$ ?

- (a)  $2^{33}$
- (b)  $2^{34}$
- (c)  $2^{35}$
- (d)  $2^{37} - 2^{35} - 2^{36}$

4. Let  $S$  be a set of size 37, and let  $x$ ,  $y$ , and  $z$  be three distinct elements of  $S$ . How many subsets of  $S$  are there that contain  $x$  or  $y$ , but do not contain  $z$ ?

- (a)  $2^{36} - 2^{34}$
- (b)  $2^{36} - 2^{35}$
- (c)  $2^{37} - 2^{34}$
- (d)  $2^{37} - 2^{35}$

5. A password consists of 12 or 13 characters, each character being one of the 10 digits  $0, 1, 2, \dots, 9$ . A password must contain the digit 7 at least once. How many passwords are there?
- (a)  $10^{12} + 10^{13} - 9^{12} - 9^{13}$   
 (b)  $12^{10} + 13^{10} - 12^9 - 13^9$   
 (c)  $10^{12} + 10^{13} - 7^{12} - 7^{13}$   
 (d)  $12^{10} + 13^{10} - 12^7 - 13^7$
6. Let  $n \geq 7$  and  $k \geq 1$  be integers, let  $A$  be the set of all bitstrings of length  $n$  that contain exactly seven 0s, and let  $B$  be the set of all bitstrings of length  $k$  that contain at least one 1. Assume there exists a one-to-one function  $f : A \rightarrow B$ . Which of the following is true?
- (a)  $2^k - 1 < \binom{n}{7}$   
 (b)  $2^k - 1 \geq \binom{n}{7}$   
 (c)  $2^k - 1 < 2^n / \binom{n}{n-7}$   
 (d)  $2^k - 1 \geq 2^n / \binom{n}{n-7}$
7. What is the coefficient of  $x^9 y^{16}$  in the expansion of  $(7x + 21y)^{25}$ ?
- (a)  $\binom{25}{16} 7^{16} 21^9$   
 (b)  $\binom{16}{25} 7^9 21^{16}$   
 (c)  $\binom{25}{16} 7^{25} 3^{16}$   
 (d) none of the above
8. How many solutions are there to the equation  $x_1 + x_2 + x_3 = 17$ , where  $x_1 \geq 0$ ,  $x_2 \geq 0$ , and  $x_3 \geq 0$  are integers?
- (a)  $\binom{19}{16}$   
 (b)  $\binom{19}{17}$   
 (c)  $\binom{20}{16}$   
 (d)  $\binom{20}{17}$

9. How many strings can be obtained by rearranging the letters of the word

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- (a)  $13!$
- (b)  $\binom{13}{4} \binom{9}{2} \binom{7}{2} \binom{5}{3}$
- (c)  $\binom{13}{4} \binom{9}{3} \binom{6}{2} \binom{4}{2}$
- (d)  $\binom{13}{1} \binom{12}{4} \binom{8}{2} \binom{6}{1} \binom{5}{2} \binom{3}{3}$

10. The function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is defined by

$$\begin{aligned} f(0) &= 2 \\ f(n+1) &= f(n) + 6n - 2 \text{ for } n \geq 0 \end{aligned}$$

What is  $f(n)$ ?

- (a)  $f(n) = 3n^2 - 5n + 2$
- (b)  $f(n) = 3n^2 + 5n + 2$
- (c)  $f(n) = 2n^2 - 5n + 2$
- (d)  $f(n) = 2n^2 + 5n + 2$

11. Consider the following recursive algorithm FIB, which takes as input an integer  $n \geq 0$ :

```
Algorithm FIB( $n$ ):  
if  $n = 0$  or  $n = 1$   
  then  $f = n$   
  else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
  endif;  
return  $f$ 
```

When running FIB(7), how many calls are there to FIB(3)?

- (a) 4
- (b) 5
- (c) 6
- (d) 7

12. The Fibonacci numbers are defined as follows:  $f_0 = 0$ ,  $f_1 = 1$ , and  $f_n = f_{n-1} + f_{n-2}$  for  $n \geq 2$ .

Consider again the recursive algorithm FIB, which takes as input an integer  $n \geq 0$ :

```
Algorithm FIB( $n$ ):  
  if  $n = 0$  or  $n = 1$   
    then  $f = n$   
    else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
    endif;  
  return  $f$ 
```

For  $n \geq 3$ , run algorithm FIB( $n$ ) and let  $a_n$  be the number of times that FIB(2) is called. Which of the following is true?

- (a) For  $n \geq 3$ ,  $a_n = f_{n-1}$
  - (b) For  $n \geq 3$ ,  $a_n = f_n$
  - (c) For  $n \geq 3$ ,  $a_n = f_{n+1}$
  - (d) For  $n \geq 3$ ,  $a_n = -1 + f_n$
13. Let  $B_n$  be the number of bitstrings of length  $n$  that do not contain 111. Which of the following is true?

- (a)  $B_n = B_{n-1} + B_{n-2} + 2^{n-3}$
- (b)  $B_n = B_{n-1} + B_{n-2} + 2^{n-3} - B_{n-3}$
- (c)  $B_n = B_{n-1} + B_{n-2} + B_{n-3}$
- (d)  $B_n = B_{n-1} + B_{n-2} + B_{n-3} + 2^{n-4}$

14. A standard deck of 52 cards has 4 Kings. Consider a hand of 9 cards, chosen uniformly at random. What is the probability that there are exactly two Kings in this hand?

- (a)  $1 - \binom{48}{7} / \binom{52}{9}$
- (b)  $\{\binom{4}{2} + \binom{48}{7}\} / \binom{52}{9}$
- (c)  $\binom{52}{9} / \{\binom{4}{2} \binom{48}{7}\}$
- (d)  $\binom{4}{2} \binom{48}{7} / \binom{52}{9}$

15. We choose a bitstring of length 25 uniformly at random. What is the probability that this string contains at least two 1s?
- (a)  $1 - (1/2)^{25} - 25(1/2)^{25}$
  - (b)  $1 + (1/2)^{25} - 25(1/2)^{25}$
  - (c)  $\sum_{k=2}^{25} \binom{25}{k} (1/2)^k$
  - (d) none of the above
16. Consider three people, each one having a uniformly random birthday (out of 365 days; we ignore leap years). What is the probability that at least two of them have the same birthday?
- (a)  $1 - \frac{365^2}{364 \cdot 363}$
  - (b)  $1 - \frac{364 \cdot 363}{365^2}$
  - (c)  $1 - \binom{3}{2}/365^3$
  - (d)  $1 - \{\binom{3}{2} + \binom{3}{3}\}/365^3$
17. What is Simon Pratt's favorite drink?
- (a) Herbal tea
  - (b) India Pale Ale
  - (c) Poutine
  - (d) None of the above, because Simon doesn't like beer



